(1) **Problem 1 (50pts)**

Verify that, for the following circuit,

![Circuit Diagram]

the maximum power transfer occurs when the following are satisfied:

\[ X_L = -X_{Th} \quad R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2} \]

where

\[ Z_L = R_L + jX_L \quad Z_{Th} = R_{Th} + jX_{Th} \]

As shown in class, using Kirchhoff’s Voltage Law:

\[ \vec{V}_{Th} = \vec{I} \cdot (Z_{Th} + Z_L) = \vec{I} \cdot [(R_{Th} + R_L) + j(X_{Th} + X_L)] \]

so the average power delivered to the load is:

\[
P = \left| \vec{I} \right|^2 R_L = \frac{\left| \vec{V}_{Th} \right|^2 R_L}{\left[ (R_{Th} + R_L)^2 + (X_{Th} + X_L)^2 \right]} \]

The maximum power transfer to the load occurs when the following two conditions are satisfied:

\[
\frac{\partial P}{\partial X_L} = 0 \quad \frac{\partial P}{\partial R_L} = 0
\]
Using the first partial derivative, and remembering:

\[
\frac{\partial}{\partial x} \left( \frac{1}{v} \right) = -\frac{1}{v^2} \frac{\partial v}{\partial x}
\]

then

\[
\frac{\partial P}{\partial X_L} = 0 = \frac{\vec{V}_{th} \cdot R_L}{\left[ (R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right]^2} \cdot 2(X_{th} + X_L)
\]

Setting the numerator to zero (the denominator is irrelevant in this calculation) we get:

\[
2(X_{th} + X_L) = 0 \quad \text{or} \quad X_L = -X_{th}
\]

Using the second partial derivative, and remembering:

\[
\frac{\partial}{\partial x} \left( \frac{u}{v} \right) = \frac{\frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2}
\]

then

\[
\frac{\partial P}{\partial R_L} = \frac{1}{\left[ (R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right]^2 \cdot \left[ (R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right]} \cdot \left[ (R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right] V_{th}^2 - V_{th}^2 \left[ R_L \cdot 2(R_{th} + R_L) \right]
\]

Once again, setting the numerator equal to zero, and after expanding terms:

\[
0 = \left[ (R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right] V_{th}^2 - V_{th}^2 \left[ R_L \cdot 2(R_{th} + R_L) \right] =
\]

\[
= R_{th}^2 + 2R_{th}R_L + R_L^2 - 2R_{th}R_L - 2R_L^2 + (X_{th} + X_L)^2
\]

\[
= R_{th}^2 - R_L^2 + (X_{th} + X_L)^2
\]

or

\[
R_L = \sqrt{R_{th}^2 + (X_{th} + X_L)^2} \quad \text{QED}
\]
(2) Problem 2 (50pts)

For the following circuit, find the average power supplied by the voltage source and the average power absorbed by the resistor and the capacitor.

The current (in phasor form) is given by:

$$\bar{I} = \frac{\bar{V}}{Z} = \frac{5\angle 30^\circ}{4 - j2} \text{ A}$$

Using the Euler Identity:

$$Z = R + jX = \sqrt{R^2 + X^2} \angle \arctan\left(\frac{X}{R}\right)$$

or

$$Z = 4 + j(-2) = \sqrt{4^2 + (-2)^2} \angle \arctan\left(\frac{-2}{4}\right) = 4.47\angle(-26.6^\circ) \text{ } \Omega$$

Therefore the current is given by:

$$\bar{I} = \frac{\bar{V}}{Z} = \frac{5\angle 30^\circ}{4.47\angle(-26.6^\circ)} = 1.12\angle(56.6^\circ) \text{ A}$$

The average power supplied by the voltage source is given by:
The average power absorbed by the resistor is given by:

\[ P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \]

or

\[ P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} (5)(1.18)\cos(30^\circ - 56.6^\circ) = 2.5\text{W} \]

The average power absorbed by the resistor is given by:

\[ P_R = I_R V_R = I_R (4I_R) = (1.18)(4 \cdot 1.18)\cos(56.6 - 56.6) = 2.5\text{W} \]

So as expected, the average power supplied by the voltage source equals the average power dissipated by the resistor, and the average power absorbed by the capacitor is zero.